

AN AUTOCORRELATION AND DISCRETE SPECTRUM FOR DYNAMICAL SYSTEMS ON METRIC SPACES

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ABSTRACT. We study dynamical systems (X, G, m) with a compact metric space X and a locally compact, σ -compact, abelian group G . We show that such a system has discrete spectrum if and only if a certain space average over the metric is a Bohr almost periodic function. In this way, this average over the metric plays for general dynamical systems a similar role as the autocorrelation measure plays in the study of aperiodic order for special dynamical systems based on point sets.

INTRODUCTION

Dynamical systems with discrete spectrum have met substantial interest in the past. In particular, they have attracted quite some attention in recent years.

One reason is that such systems play an important role in the investigation of aperiodic order. Aperiodic order, also known as mathematical theory of quasicrystals, has emerged as a substantial topic of research over the last three decades, see e.g. [1] for a recent monograph and [11] for a recent collection of surveys. A key feature of aperiodic order is the occurrence of (pure) point diffraction. Due to a collective effort over the years pure point diffraction is now understood as discrete spectrum of suitable associated dynamical systems [5, 12, 10, 3, 16, 15].

Another instance of the recent interest in discrete spectra can be found in a series of works which analyse such spectra via weak notions of equicontinuity, [7, 8, 9]. These works provide in particular a characterization of discrete spectrum (see [6, 23] for related work as well) and a characterization of discrete spectrum with continuous eigenfunctions and unique ergodicity.

The dynamical systems (X, G, m) underlying the investigation of aperiodic order (and defined in detail in Section 5) have a special structure. The compact space X , on which the locally compact, σ -compact, abelian group G acts, consists of point sets or more generally measures. Accordingly, these systems are known as *translation bounded measure dynamical systems* (TMDS). The fact that the points of X are measures allows one to pair elements of X with elements from the vector space $C_c(G)$ of continuous compactly supported functions on the group resulting in a map N from $C_c(G)$ to functions on X . Via this map one can then define the *autocorrelation measure* γ associated to (X, G, m) . The Fourier transform of the autocorrelation measure is known as *diffraction measure*. The diffraction measure or, equivalently, the autocorrelation measure encodes a remarkable amount of information on the original system. In fact, a main result of the

theory (already mentioned above and discussed in Section 5 in more detail) can be stated as follows:

Result - TMDS. [12, 10, 3, 16, 15] *The TMDS (X, G, m) has discrete spectrum if and only if the measure γ is strongly almost periodic. In this case, the group generated by the frequencies of γ is the group of eigenvalues of (X, G, m) .*

In a general dynamical system (X, G, m) (again with notation to be explained in detail later in Section 1) the points of X can not be paired with elements of $C_c(G)$. Hence, such a system does not admit an autocorrelation. However, if d is a metric on X inducing the topology, then - as we will show below - the function

$$\underline{d} : G \longrightarrow [0, \infty), \underline{d}(t) = \int_X d(x, tx) dm(x),$$

can serve as a convenient analogue to the autocorrelation. Indeed, our main abstract result reads as follows:

Main result. (Compare Theorem 4.1 below.) *The dynamical system (X, G, m) has discrete spectrum if and only if \underline{d} is almost periodic in the sense of Bohr. In this case, the group of eigenvalues is generated by the frequencies of \underline{d} .*

For general dynamical systems over metric spaces, this result provides an analogue to the result above for TMDS. As a consequence we also obtain (in Corollary 4.3) a converse to a result of [21]. This is particular remarkable as it is mentioned in [21] that 'it is unlikely' that such a converse hold. Finally - as it is to be expected - our considerations allow us to also reprove the above result for TMDS provided the group G is metrizable, see Section 5.

Our approach to discrete spectrum is somewhat complementary to the approach in the quoted works [7, 8, 9, 23] above. A central quantity in these works is the pseudometric \bar{d} on X , which arises by averaging d over G ,

$$\bar{d}(x, y) = \limsup_{n \rightarrow \infty} \frac{1}{m_G(F_n)} \int_{F_n} d(tx, ty) dm_G(t),$$

where (F_n) is a Følner sequence and m_G denotes Haar measure on G . The discreteness of the spectrum (and related phenomena) is then encoded in equicontinuity and covering properties with respect to the topology induced by \bar{d} . In comparison our key quantity \underline{d} can (essentially) be seen as a metric on G , which arises by averaging over X . Discreteness of the spectrum is then encoded in almost periodicity properties of \underline{d} . These two points of view are certainly related. For example, it is possible to use our result to reprove parts of the abstract considerations of [9] on spectral isomorphy. This and more will be considered elsewhere.

The necessary notation and set-up for our investigations is provided in Section 1. The main technical tools are gathered in Section 2. There we also introduce the domination relation \prec which is a key ingredient in our analysis. With these tools we can then provide various characterizations of almost periodicity of \underline{d} in Section 3. The main result, given in Section 4, is then a rather direct consequence of these characterizations.

1. PRELIMINARIES

In this section we set up notation and recall a few standard facts on dynamical systems. The material is well-known.

We consider a compact space X equipped with a continuous action

$$G \times X \longrightarrow X, (t, x) \mapsto tx,$$

of the locally compact, σ -compact, abelian group G and a probability measure m , which is invariant under the action of G . We then call (X, G, m) a *dynamical system over the space X* . Throughout we will assume that the topology of X is induced by a metric. This metric will usually be denoted by d .

The action of G on X induces unitary operators $T_t : L^2(X, m) \longrightarrow L^2(X, m)$ with

$$T_t f = f(t \cdot)$$

for each $t \in G$. Here, $L^2(X, m)$ is the Hilbert space of (equivalence classes of) square integrable functions on X . It is equipped with the inner product

$$\langle f, g \rangle = \int_X \bar{f} g \, dm$$

and the associated norm

$$\|f\| := \sqrt{\langle f, f \rangle}$$

for $f, g \in L^2(X, m)$. We will be particularly interested in continuous functions on X and will denote the vector space of all such functions by $C(X)$.

An $f \in L^2(X, m)$ with $f \neq 0$ is called an *eigenfunction to the eigenvalue* $\gamma \in \widehat{G}$ if $T_t f = \gamma(t)f$ holds for each $t \in G$. Here, \widehat{G} is the *dual group of G* consisting of all continuous group homomorphisms $\gamma : G \longrightarrow \{z \in \mathbb{C} : |z| = 1\}$ equipped with multiplication of functions and complex conjugation as product and inverse respectively. We denote the group generated by the set of eigenvalues as the *group of eigenvalues*. If the dynamical system is minimal (i.e. each orbit is dense) or ergodic (i.e. any measurable invariant set has measure 0 or 1), then the set of eigenvalues forms already a group.

The dynamical system (X, G, m) is said to have *discrete spectrum* if there exists an orthonormal basis of $L^2(X, m)$ consisting of eigenfunctions.

Whenever f is a bounded function from a set Y to the complex numbers, we define the *supremum norm* of f as $\|f\|_\infty := \sup\{|f(y)| : y \in Y\}$. We will be interested in the cases $Y = G$ and $Y = X$.

We write the group operation on G additively and denote the neutral element of G by 0.

2. THE FUNCTIONS \underline{e} AND e'

Let (X, G, m) be a dynamical system over a compact space X . A *pseudometric* on a set Y is a function $e : Y \times Y \longrightarrow [0, \infty)$ satisfying $e(x, x) = 0$, $e(x, y) = e(y, x)$ and $e(x, y) \leq e(x, z) + e(z, y)$ for all $x, y, z \in Y$. We will be interested in pseudometrics on X and on G .

To a continuous pseudometric e on X we associate the functions

$$\underline{e} : G \longrightarrow [0, \infty), \quad \underline{e}(t) := \int_X e(x, tx) dm(x),$$

and

$$e' : G \times G \longrightarrow [0, \infty), \quad e'(s, t) := \int_X e(sx, tx) dm(x).$$

A short computation (using the invariance of m) gives

$$e'(s, t) = \underline{e}(s - t) \text{ and } \underline{e}(s) = e'(0, s).$$

For this reason properties of \underline{e} and of e' are strongly connected and it usually suffices to study one of these functions. A few basic properties of e' are gathered next.

Proposition 2.1 (Basic properties of e'). *The function e' is a continuous, bounded and G -invariant pseudometric.*

Proof. Continuity of e' is clear from continuity of the group action and compactness of X . Boundedness of e' follows as m is a probability measure. The G -invariance is clear from the formula $e'(s, t) = \underline{e}(s - t)$. It remains to show that e' is a pseudometric. This follows easily as e is a pseudometric. \square

Lemma 2.2 (Functions inducing invariant metrics). *Let $F : G \longrightarrow [0, \infty)$ be a function on G such that $F'(t, s) := F(t - s)$ is a pseudometric. Then, F satisfies $F(0) = 0$, $F(-s) = F(s)$ and $|F(s) - F(t)| \leq F(s - t)$ and*

$$\|F(s + \cdot) - F(t + \cdot)\|_\infty = F(s - t)$$

for all $s, t \in G$. If F is continuous (at $t = 0$) it is uniformly continuous on G .

Proof. As F' is a pseudometric we infer $F(0) = F'(0, 0) = 0$ and $F(s) = F'(s, 0) = F'(0, s) = F(-s)$ as well as

$$|F(s) - F(t)| = |F'(s, 0) - F'(t, 0)| \leq F'(t, s) = F(t - s).$$

for all $s, t \in G$. From this inequality we directly obtain the inequality $\|F(s + \cdot) - F(t + \cdot)\|_\infty \leq F(s - t)$ for all $s, t \in G$. The reverse inequality \geq follows by inserting the value $-t$. Now, the last statement is clear. \square

Proposition 2.3 (Basic properties of \underline{e}). *Let e be a continuous pseudometric on X . The function \underline{e} is bounded and satisfies $\underline{e}(0) = 0$, $\underline{e}(s) = \underline{e}(-s)$ as well as $|\underline{e}(s) - \underline{e}(t)| \leq \underline{e}(s - t)$ and*

$$\|\underline{e}(\cdot + s) - \underline{e}(\cdot + t)\|_\infty = \underline{e}(s - t)$$

for all $s, t \in G$. Moreover, \underline{e} is uniformly continuous.

Proof. As X is compact and e is continuous, the function \underline{e} is bounded and continuous. As for the remaining statements we can apply the previous lemma with $F = \underline{e}$ and $F' = e'$. \square

Of course, the functions \underline{e} and e' depend on e . So, one may wonder how they change if e is replaced by another pseudometric. To investigate this we introduce the following concept.

Definition 2.4 (The relation \prec). Let f and g be functions from a set Y to the complex numbers. Then, f is said to *dominate* g written as $g \prec f$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|g(y)| \leq \varepsilon$ whenever $|f(y)| \leq \delta$ holds. If both f dominates g and g dominates f we say that f and g are equivalent and write $f \sim g$.

The following statement is a rather direct consequence of uniform continuity of continuous functions on compact sets.

Proposition 2.5. *Let e be a pseudometric on the compact X . Let d be a metric on X inducing the topology. Then, e is continuous if and only if it is dominated by d .*

By the previous proposition, two metrics d and e on the compact X giving the topology are equivalent.

Lemma 2.6. *Let (X, G, m) be a dynamical system. Let e_1 and e_2 be continuous pseudometrics on X with $e_1 \prec e_2$. Then,*

$$\underline{e_1} \prec \underline{e_2} \text{ and } e'_1 \prec e'_2.$$

Proof. It suffices to show the statement for $\underline{e_1}$ and $\underline{e_2}$. We have to show that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\underline{e_2}(t) < \delta$ (for $t \in G$) implies $\underline{e_1}(t) < \varepsilon$. So, let $\varepsilon > 0$ be given. Without loss of generality we can assume $e_1, e_2 \leq 1$.

By $e_1 \prec e_2$, there exists $\varepsilon_1 > 0$ with

$$e_2(x, y) \leq \varepsilon_1 \implies e_1(x, y) \leq \frac{\varepsilon}{2}.$$

Without loss of generality we can assume

$$\varepsilon_1 \leq \frac{\varepsilon}{2}.$$

Set $\delta := \varepsilon_1^2$ and consider a $t \in G$ with $\underline{e_2}(t) \leq \delta$. Setting

$$M := \{x : e_2(x, tx) \geq \varepsilon_1\}$$

we then find

$$\varepsilon_1 m(M) \leq \int_X e_2(x, tx) dm(x) = \underline{e_2}(t) < \varepsilon_1^2.$$

This gives

$$m(M) \leq \varepsilon_1.$$

By construction we have $e_2(x, tx) < \varepsilon_1$ and hence $e_1(x, tx) \leq \frac{\varepsilon}{2}$ for $x \in X \setminus M$. Given this a short computation shows

$$\begin{aligned} \underline{e_1}(t) &= \int_X e_1(x, tx) dm(x) \\ &= \int_M e_1(x, tx) dm(x) + \int_{X \setminus M} e_1(x, tx) dm(x) \\ &\leq \|e_1\|_\infty m(M) + \frac{\varepsilon}{2} m(X \setminus M) \\ &\leq \varepsilon_1 + \frac{\varepsilon}{2} \\ &\leq \varepsilon. \end{aligned}$$

This finishes the proof. □

For us certain (pseudo)metrics will be of special interest: To each continuous $f : X \rightarrow \mathbb{C}$ we associate the pseudometric

$$e_f : X \times X \rightarrow [0, \infty), e_f(x, y) := |f(x) - f(y)|.$$

As f is continuous, so is e_f . In particular, e_f is dominated by the metric d . Let now for $n \in \mathbb{N}$ functions $f_n \in C(X)$ and $c_n \geq 0$ with $\sum c_n \|f_n\|_\infty < \infty$ be given. Then,

$$e_{(f_n), (c_n)} := \sum_n c_n e_{f_n}$$

is a continuous pseudometric. To an $f \in C(X)$ we can also associate the function

$$F_f : G \rightarrow [0, \infty), F_f(t) := \|f - T_t f\|.$$

To $f_n \in C(X)$ and $c_n \geq 0$ with $\sum c_n \|f_n\|_\infty < \infty$ we can moreover associate the function

$$F_{(f_n), (c_n)} := \sum_n c_n F_{f_n}.$$

Proposition 2.7. (a) Let f be a continuous function on X . Then, $\underline{e}_f \prec \underline{d}$.

(b) Let f be a continuous function on X . Then, $F_f \leq \sqrt{2\|f\|_\infty} \cdot \sqrt{\underline{e}_f}$ and $\underline{e}_f \leq F_f$ hold. In particular, $F_f \sim \underline{e}_f$ and $F_f \prec \underline{d}$.

(c) Let continuous functions f_n on X and $c_n > 0$, $n \in \mathbb{N}$, with $\sum_n c_n \|f_n\|_\infty < \infty$ be given such that the (f_n) separate the points of X . Then,

$$\underline{e}_{(f_n), (c_n)} \sim F_{(f_n), (c_n)} \sim \underline{d}.$$

Proof. (a) We have already discussed that e_f is continuous. Hence, it is dominated by d . Thus, the previous lemma gives $\underline{e}_f \prec \underline{d}$.

(b) To show the bound on F_f we compute

$$\begin{aligned} F_f(t) &= \|f - T_t f\| \\ &= \left(\int |f(x) - f(tx)|^2 dm(x) \right)^{1/2} \\ &\leq \sqrt{2\|f\|_\infty} \left(\int |f(x) - f(tx)| dm(x) \right)^{1/2} \\ &= \sqrt{2\|f\|_\infty} \cdot \sqrt{\underline{e}_f(t)}. \end{aligned}$$

To show the bound on \underline{e}_f , we note that m is a probability measure, and hence Cauchy-Schwarz inequality gives $\int_X |g| dm \leq \|g\|$ for all g in $L^2(X, m)$. Thus, we can estimate

$$\begin{aligned} \underline{e}_f(t) &= \int_X e_f(x, tx) dm(x) \\ &= \int |f(x) - f(tx)| dm(x) \\ &\leq \|f - T_t f\| \\ &= F_f(t) \end{aligned}$$

for each $t \in G$.

The preceding two bounds give $F_f \sim \underline{e}_f$. Invoking (a), we then also infer $F_f \prec \underline{d}$.

(c) From (b) and the summability condition on the sequence (c_n) we directly infer $\underline{e}_{(f_n), (c_n)} \leq F_{(f_n), (c_n)}$ as well as

$$F_{(f_n), (c_n)} \leq \sum_n c_n \sqrt{2\|f_n\|_\infty} \sqrt{\underline{e}_{f_n}} \leq \left(2 \sum_n c_n \|f_n\|_\infty\right)^{1/2} \sqrt{\underline{e}_{(f_n), (c_n)}}.$$

This gives

$$\underline{e}_{(f_n), (c_n)} \sim F_{(f_n), (c_n)}.$$

From (a) and the summability condition on the sequence (c_n) we also easily find $\underline{e}_{(f_n), (c_n)} \prec \underline{d}$. It remains to show $\underline{d} \prec \underline{e}_{(f_n), (c_n)}$. Now, by the assumptions the function $e := e_{(f_n), (c_n)}$ is a continuous metric. As X is a compact Hausdorff space this metric must then generate the topology as well. Thus, d is continuous with respect to the metric e and vice versa. Thus, d and e are equivalent. Thus, the previous lemma gives $\underline{d} \sim \underline{e}$. This finishes the proof. \square

3. ALMOST PERIODICITY AND \underline{d}

Recall that a subset \mathcal{T} of G is called *relatively dense* if there exists a compact $K \subset G$ with Minkowski sum

$$\mathcal{T} + K := \{\tau + k : \tau \in \mathcal{T}, k \in K\}$$

equal to G . A uniformly continuous function $F : G \rightarrow \mathbb{C}$ is called *almost periodic (in the sense of Bohr)* if for any $\varepsilon > 0$ the set

$$\{t \in G : \|F(t + \cdot) - F\|_\infty \leq \varepsilon\}$$

is relatively dense. This is the case if and only if the *hull of F* defined by

$$\mathbb{T}(F) := \overline{\{F(t + \cdot) : t \in G\}}^{\|\cdot\|_\infty}$$

is compact. In this case the hull has a unique group structure making it into a topological group such that

$$j : G \rightarrow \mathbb{T}(F), \quad t \mapsto F(t + \cdot),$$

is a continuous group homomorphism. Clearly, the map j has dense range. Hence, the group $\mathbb{T}(F)$ must be abelian. Moreover, the dual map

$$\widehat{\mathbb{T}(F)} \rightarrow \widehat{G}, \quad \gamma \mapsto \gamma \circ j,$$

is injective. Hence, any element $\gamma \in \widehat{\mathbb{T}(F)}$ can be considered as an element of \widehat{G} and this is how we will think about elements of $\widehat{\mathbb{T}(F)}$.

As $\mathbb{T}(F)$ is a compact group it carries a unique normalized invariant measure $m_{\mathbb{T}(F)}$ and the elements of $\widehat{\mathbb{T}(F)}$ form an orthonormal basis in the Hilbert space $L^2(\mathbb{T}(F), m_{\mathbb{T}(F)})$. Consequently any continuous function h on $\mathbb{T}(F)$ can then be expanded uniquely in a *Fourier series*

$$h = \sum_{\gamma \in \widehat{\mathbb{T}(F)}} c_\gamma^h \gamma,$$

where the sum converges in the L^2 -sense. Whenever H is a continuous function on G such that there exists a (necessarily unique) continuous h on $\mathbb{T}(F)$ with $H = h \circ j$ we call

$$\{\gamma \circ j : c_\gamma^h \neq 0\} \subset \widehat{G}$$

the *set of frequencies of H* . Clearly, one such function is given by $H = \delta \circ j$ with the function $\delta : \mathbb{T}(F) \rightarrow \mathbb{C}, \delta(E) = E(0)$. In this case the group generated by the frequencies of H is just $\widehat{\mathbb{T}(F)}$.

We will be interested in proving almost periodicity of functions such as \underline{e} for a continuous pseudometric e on X , F_f for $f \in C(X)$ and $F_{(c_n), (f_n)}$ for $f_n \in C(X)$ and $c_n \geq 0$ with $\sum_n c_n \|f_n\| < \infty$. All of these are functions $F : G \rightarrow [0, \infty)$ such that F' with $F'(s, t) := F(s - t)$ is a continuous pseudometric on G . Hence, they are continuous $F : G \rightarrow [0, \infty)$ with $F(0) = 0$ and

$$\|F(s + \cdot) - F(t + \cdot)\|_\infty = F(s - t) \quad (1)$$

for all $s, t \in G$ (compare Lemma 2.2). For such functions we clearly have the following criterion for almost periodicity.

Lemma 3.1. *A continuous $F : G \rightarrow [0, \infty)$ with $F(0) = 0$ and $\|F(s + \cdot) - F_j(t + \cdot)\|_\infty = F(s - t)$ for all $s, t \in G$ is almost periodic if and only if for every $\varepsilon > 0$ the set*

$$\{\tau \in G : |F(\tau)| \leq \varepsilon\} \quad (2)$$

is relatively dense.

Accordingly, proving relative denseness of sets as in (2) will be our main tool in dealing with almost-periodicity.

As relative denseness of sets as in (2) is clearly preserved under \prec we obtain the following consequence.

Proposition 3.2. *Let $F_j : G \rightarrow [0, \infty)$, $j \in \{1, 2\}$, be continuous with $F_j(0) = 0$ and*

$$\|F_j(s + \cdot) - F_j(t + \cdot)\|_\infty = F_j(s - t)$$

for all $s, t \in G$ and $F_1 \prec F_2$. Then, F_1 is almost periodic if F_2 is almost periodic.

Proof. By $F_1 \prec F_2$ relative denseness of $\{\tau \in G : |F_2(\tau)| \leq \varepsilon\}$ for each $\varepsilon > 0$ implies relative denseness of $\{\tau \in G : |F_1(\tau)| \leq \delta\}$ for each $\delta > 0$. Now, the proof follows from Lemma 3.1. \square

The following lemma gathers various equivalent versions of almost periodicity of \underline{d} . Our main result will be a rather direct consequence of this lemma. The equivalence between (v), (v') and (vi) is well-known. As we will need this equivalence, we include a discussion for completeness reasons.

Lemma 3.3. *Let (X, G, m) be a dynamical system. Then, the following assertions are equivalent:*

- (i) *The function \underline{d} is almost periodic.*
- (ii) *For each $f \in C(X)$ the function \underline{e}_f is almost periodic.*
- (ii)' *For each $f \in C(X)$ the function F_f is almost periodic.*

- (iii) The set of $f \in C(X)$ for which \underline{e}_f is almost periodic separates the points of X .
- (iii)' The set of $f \in C(X)$ for which F_f is almost periodic separates the points of X .
- (iv) The function $\underline{e}_{(f_n), (c_n)}$ is almost periodic for one (each) set of functions f_n in $C(X)$ and $c_n > 0$, $n \in \mathbb{N}$, with $\sum_n c_n \|f_n\| < \infty$ such that the (f_n) separate the points of X .
- (iv)' The function $F_{(f_n), (c_n)}$ is almost periodic. for one (each) set of functions f_n in $C(X)$ and $c_n > 0$, $n \in \mathbb{N}$, with $\sum_n c_n \|f_n\| < \infty$ such that the (f_n) separate the points of X .
- (v) For any $f \in C(X)$ the function $G \rightarrow \mathbb{C}, t \mapsto \langle f, T_t f \rangle$ is almost periodic.
- (v)' For any $f \in L^2(X, m)$ the function $G \rightarrow \mathbb{C}, t \mapsto \langle f, T_t f \rangle$ is almost periodic.
- (vi) The set of $f \in C(X)$ for which $G \rightarrow \mathbb{C}, t \mapsto \langle f, T_t f \rangle$ is almost periodic separates the points of X .

If one of these equivalent conditions holds the group generated by the frequencies of \underline{d} is the same as the group generated by the frequencies of all functions of the form $G \rightarrow \mathbb{C}, t \mapsto \langle f, T_t f \rangle$ for $f \in L^2(X, m)$.

Proof. We first discuss the equivalence of the various statements: Note that the functions in question in statements (i) to (iv)' are continuous, vanish at the neutral element of G and satisfy the assumption (1). Thus, we can invoke the previous proposition to study their almost-periodicity. Given this, the equivalence between statements (i) to (iv)' is a direct consequence of Proposition 2.7. Note that (iii) implies (iv) and (iii)' implies (iv)' as each set of continuous functions that separates the points must contain a countable subset of functions which also separates the points of X (due to compactness of X).

The implications (v)' \implies (v) \implies (vi) are clear. The crucial ingredient for the remaining part of the proof is the equality

$$F_f^2(t) = 2\|f\|^2 - \langle f, T_t f \rangle - \overline{\langle f, T_t f \rangle}, \quad (3)$$

which follows by a direct computation.

Given this equality the implication (vi) \implies (iii)' follows immediately. Conversely, (ii)' implies almost periodicity of F_f for any real-valued $f \in C(X)$ and the above equality (3) then gives almost periodicity of $\langle f, T_t f \rangle$ for any such f . By a simple polarisation argument this then gives almost periodicity of $\langle f, T_t g \rangle$ for all real valued continuous f, g . This, in turn easily implies (v) from which (v)' follows by denseness of $C(X)$ in $L^2(X, m)$.

We now turn to the last statement: Set $S_f : G \rightarrow \mathbb{C}, S_f(t) = \langle f, T_t f \rangle$ for $f \in L^2(X, m)$. Whenever $f \in C(X)$ is real valued, we have $F_f^2 = 2\|f\|^2 - 2S_f$ by (3). Hence, we can lift S_f to a continuous function on $\mathbb{T}(F_f)$. As $F_f \prec \underline{d}$ due to Proposition 2.7 we also have $F_f' \prec d'$. This implies that there exists a continuous map $\pi : \mathbb{T}(\underline{d}) \rightarrow \mathbb{T}(F_f)$ mapping $\underline{d}(t + \cdot)$ to $F_f(t + \cdot)$ for any $t \in G$. Hence, we can then lift S_f to a continuous function on $\mathbb{T}(\underline{d})$ as well. By considering real- and imaginary parts we can then lift S_f to a continuous function on $\mathbb{T}(\underline{d})$ for any $f \in C(X)$. Taking limits and using that $C(X)$ is

dense in $L^2(X, m)$ we can then lift S_f for any $f \in L^2(X, m)$ to $\mathbb{T}(\underline{d})$. Hence, the set of frequencies of S_f is contained in the set of frequencies of \underline{d} for any $f \in L^2(X, m)$.

Conversely, consider $F := F_{(f_n), (c_n)}$ for a set of functions f_n in $C(X)$ and $c_n > 0$, $n \in \mathbb{N}$, with $\sum_n c_n \|f_n\| < \infty$ such that the (f_n) separate the points of X . Then, $F \sim \underline{d}$. Hence, $\mathbb{T}(F) = \mathbb{T}(\underline{d})$. Thus, the frequencies of \underline{d} agree with the frequencies of F , which in turn are contained in the group generated by the frequencies associated to the f_n , $n \in \mathbb{N}$. \square

4. A CHARACTERIZATION OF DISCRETE SPECTRUM

In this section we state and prove our main result which gives a characterization of dynamical systems with discrete spectrum.

Theorem 4.1 (Characterizing discrete spectrum). *Let (X, G, m) be a dynamical system. Let d be a metric on X inducing the topology. Then, the following assertions are equivalent:*

- (i) *The dynamical system (X, G, m) has discrete spectrum.*
- (ii) *The function \underline{d} is almost periodic.*

If one of these equivalent conditions hold the group of eigenvalues of (X, G, m) equals the group generated by the frequencies of \underline{d} .

Remark. If \underline{d} is almost periodic, the group generated its frequencies is the dual group of $\mathbb{T}(\underline{d})$ (as follows from the general discussion above). Moreover, it is not hard to see that this group is also given as the Hausdorff completion of G with respect to d' .

Proof. We first deal with the equivalence statement. By the Lemma 3.3, the function \underline{d} is almost periodic if and only if $G \rightarrow \mathbb{C}$, $t \mapsto \langle f, T_t f \rangle = S_f(t)$ is almost periodic for any $f \in L^2(X, m)$. This in turn is a well-known characterization of discrete spectrum. Indeed, for any $f \in L^2(X, m)$ there exists a unique measure μ_f on \widehat{G} with

$$\langle f, T_t f \rangle = \int_{\widehat{G}} \gamma(t) d\mu_f(\gamma)$$

(see e.g. [17]). Now, discrete spectrum just means that all these measures are point measures and pure pointedness of μ_f is equivalent to almost periodicity of $t \mapsto \langle f, T_t f \rangle$ (see e.g. [16] for recent discussion).

We now turn to the second statement of the theorem. By the last statement of Lemma 3.3 the group generated by the frequencies of \underline{d} is the group generated by the frequencies of the S_f , $f \in L^2(X, m)$. The frequencies of S_f , however, are just the atoms of the measures μ_f . Hence, they generate the group of eigenvalues. \square

It is possible to rephrase the result in terms of almost-periods. This will clarify the relationship between our result and earlier results. Whenever e is a continuous pseudometric on X a $t \in G$ is called a *measure theoretic ε -almost period of e* if

$$m(\{x \in X : e(x, tx) > \varepsilon\}) < \varepsilon.$$

Lemma 4.2 (Almost periodicity and measure-theoretic ε - almost periods). *Let e be a continuous pseudometric on X . Then, the following assertions are equivalent:*

- (i) *For any $\varepsilon > 0$ the set of measure theoretic ε -almost periods of e is relatively dense.*
- (ii) *The function \underline{e} is almost periodic.*

Proof. (i) \implies (ii): We have to show that the set of $\{t \in G : \underline{e}(t) \leq \varepsilon\}$ is relatively dense for any $\varepsilon > 0$ (compare Lemma 3.1). Let $\varepsilon_1 > 0$ with

$$\varepsilon_1 \|e\|_\infty + \varepsilon_1 < \varepsilon$$

be given. Chose a measure theoretic ε_1 -almost period t of e and set

$$M := \{x \in X : e(x, tx) > \varepsilon_1\}.$$

Then, a direct computation gives

$$\underline{e}(t) = \int_M e(x, tx) dm + \int_{X \setminus M} e(x, tx) dm \leq \varepsilon_1 \|e\|_\infty + m(X \setminus M) \varepsilon_1 < \varepsilon.$$

As the set of measure theoretic ε_1 -almost periods is relatively dense by (i) the desired statement follows.

(ii) \implies (i): This follows by mimicking an argument given in the proof of Lemma 2.6: Let $\varepsilon > 0$ be given. By (ii) the set $\{t \in G : \underline{e}(t) < \varepsilon^2\}$ is relatively dense. For any t in this set we obtain with $N := \{x \in X : e(x, tx) > \varepsilon\}$

$$\varepsilon m(N) \leq \int_X e(x, tx) dm = \underline{e}(t) < \varepsilon^2$$

and, hence, $m(N) < \varepsilon$. This finishes the proof. \square

From the previous lemma and the main result, Theorem 4.1, we directly obtain the following consequence.

Corollary 4.3. *Let (X, G, m) be a dynamical system and d a metric on X inducing the topology on X . Then, the following assertions are equivalent:*

- (i) *For any $\varepsilon > 0$ the set of measure theoretic ε -almost periods of d is relatively dense.*
- (ii) *The dynamical system has discrete spectrum.*

Remarks. The implication (i) \implies (ii) of the corollary is proven as Theorem 3.2 in [21] (under an additional ergodicity assumption). There also a partial converse is provided in Proposition 3.3. This converse needs an additional requirement of continuity of eigenfunctions and it is remarked that 'it is unlikely' that a full converse of the theorem holds. Our corollary shows that such a full converse does hold.

We can also derive the following consequence.

Corollary 4.4. *Let (X, G, m) be a dynamical system. Let \mathcal{F} be a family of continuous functions on X , which separates the points of X . Then, the following assertions are equivalent:*

- (i) *For any $\varepsilon > 0$ and each $f \in \mathcal{F}$ the set of ε -almost periods of \underline{e}_f is relatively dense.*

(ii) *The dynamical system has discrete spectrum.*

Proof. (ii) \implies (i): This follows from combining the implication (i) \implies (ii) of Theorem 4.1, the implication (i) \implies (ii) of Lemma 3.3 and the first lemma of this section. (This reasoning actually works for any $f \in C(X)$ and not only for $f \in \mathcal{F}$.)

(i) \implies (ii): This follows from combining the first lemma of this section with the implication (iii) \implies (i) of Lemma 3.3 and Theorem 4.1 (ii) \implies (i). \square

Remark. A possible choice of the family \mathcal{F} is given as $d(x, \cdot)$, $x \in X$.

5. CONNECTION TO THE AUTOCORRELATION MEASURE

Our considerations are motivated by the study of diffraction theory for quasicrystals. Diffraction theory for quasicrystals and the relationship with dynamical systems has gained substantial attention in the last two decades. Indeed, from the very beginning tiling and point set dynamical systems with discrete spectrum have played a key role [5, 19, 21, 22, 20]. We refer the interested reader to the surveys [2, 13] and the corresponding parts of [14] for further details and references. In this section we briefly sketch the necessary background to put our main result into this context.

As discussed in [3], diffraction theory for quasicrystals can conveniently be set up in the framework of translation bounded measures on a locally compact, σ -compact abelian group G . In fact, one can go beyond translation bounded measures and deal with measures satisfying only existence of a suitable second moment [16] or even deal with suitable pairings of functions on G and the elements of the dynamical system [15]. However, here we will stick to translation bounded measures as this seems to encompass the usual models for quasicrystals. We follow [3] to which we refer for further details and references.

Let $C_c(G)$ be the space of continuous functions on G with compact support. A measure μ on G is called *translation bounded* if its total variation $|\mu|$ satisfies

$$\sup |\mu|(t + U) < \infty$$

for one (all) relatively compact open U in G . The set of all translation bounded measures is denoted by $M^\infty(G)$. It is equipped with the vague topology. There is a natural action α of G on $M^\infty(G)$ by translations where for $t \in G$ and $\mu \in M^\infty(G)$ the measure $\alpha_t(\mu)$ satisfies $\alpha_t(\mu)(\varphi) = \mu(\varphi(\cdot - t))$ for all $\varphi \in C_c(G)$. Whenever X is a compact subset of $M^\infty(G)$ which is invariant under the translation action and m is an invariant probability measure on X , we call (X, G, m) a *dynamical system of translation bounded measures* or just TMDS for short. Such a system comes with a canonical map

$$N : C_c(G) \longrightarrow C(X) \text{ with } N_\varphi(\mu) = \int \varphi(-s) d\mu(s).$$

Let us emphasize that the existence of such a map is a distinctive feature of TMDS compared to general dynamical systems. There exists then a unique translation bounded measure γ on (X, G, m) with

$$\gamma * \varphi * \tilde{\varphi}(t) = \langle N_\varphi, T_t N_\varphi \rangle \tag{4}$$

for all $\varphi \in C_c(G)$ and all $t \in G$. Here, $*$ denotes the convolution and $\tilde{\varphi}(t) = \overline{\varphi(-t)}$. The measure γ is called the *autocorrelation* of the TMDS. This measure allows for a Fourier transform $\hat{\gamma}$ which is a (positive) measure on \hat{G} . A main result of the theory gives the following:

Theorem. *The TMDS (X, G, m) has discrete spectrum if and only if the measure γ is strongly almost periodic. In this case, the group of eigenvalues of (X, G, m) is the group generated by $\{k \in \hat{G} : \hat{\gamma}(\{k\}) > 0\}$.*

A few **remarks** are in order:

- The measure γ is called strongly almost periodic if $\gamma * \varphi$ is almost periodic (in the sense of Bohr) for all $\varphi \in C_c(G)$.
- The above theorem is usually formulated with the assumption that $\hat{\gamma}$ is a pure point measure. However, as is well-known, see e.g. Proposition 7 and Theorem 4 in [4] for a discussion in the context of aperiodic order, the measure γ is strongly almost periodic if and only if $\hat{\gamma}$ is a pure point measure.
- The theorem has a long history. The connection between diffraction measure and point spectrum goes back to work of Dworkin [5]. The first statement giving an equivalence (in the more restricted setting of uniquely ergodic dynamical systems of point sets satisfying the regularity requirement of finite local complexity) can be found in [12]. This was then generalized (in slightly different directions) in [10] and [3]. A unified treatment of [3, 10] was given in [16]. Recently an even more general result was given in [15]. In the context of TMDS discussed in this section the theorem was first proven in [3].

Clearly, the preceding result is quite close to our main result: Pure point-ness of the spectrum is characterized by almost periodicity of a suitable function (in this case the measure γ) and the group of eigenvalues is generated by the frequencies of the function (in this case the atoms of the diffraction measure). In fact, we can derive the previous result from our main result (provided the group is metrizable) along the following lines:

The measure γ is strongly almost periodic if and only if $\gamma * \varphi * \tilde{\varphi}$ is almost periodic for all $\varphi \in C_c(G)$. As G is metrizable and σ -compact, there exists a dense set φ_n , $n \in \mathbb{N}$, of functions in $C_c(G)$ and γ is almost periodic if and only if $\gamma * \varphi_n * \tilde{\varphi}_n$ is almost periodic for each $n \in \mathbb{N}$.

By the formula given above in (4) this is equivalent to $t \rightarrow \langle N_{\varphi_n}, T_t N_{\varphi_n} \rangle$ being almost periodic for all $n \in \mathbb{N}$. As the φ_n , $n \in \mathbb{N}$, are dense in $C_c(G)$, they clearly separate the points of X . Hence, condition (vi) of Lemma 3.3 is satisfied. The lemma then gives almost periodicity of \underline{d} and Theorem 4.1 shows discrete spectrum. The statement on the frequencies follows by going through the argument and keeping track of the involved frequencies.

Remark. In the case of TMDS the map N and the Fourier transform $\hat{\gamma}$ of the autocorrelation measure γ arise by a limiting procedure out of quantities which are defined for any dynamical system. Details are discussed in this remark: Whenever (X, G, m) is a dynamical system each $f \in L^2(X, m)$ gives rise to the map

$$N^f : C_c(G) \longrightarrow L^2(X, m)$$

via $N^f(\varphi) := \int \varphi(-t)T_t f dt$. A direct computation then shows for any $\varphi, \psi \in C_c(G)$

$$\langle N^f(\varphi), N^f(\psi) \rangle = \int_{\widehat{G}} \overline{\widehat{\varphi}} \widehat{\psi} d\mu_f(\gamma), \quad (5)$$

where $\widehat{\sigma}$ denotes the Fourier transform of σ defined by

$$\widehat{\sigma}(\gamma) = \int \overline{\gamma(t)} \sigma(t) dm_G(t)$$

and μ_f is the spectral measure of f (i.e. the unique finite measure μ_f on \widehat{G} with $\langle f, T_t f \rangle = \int_{\widehat{G}} \gamma(t) d\mu_f(\gamma)$ for all $t \in G$). In this way, we obtain for any specific f a map N^f , which encodes the spectral measure of f in the sense that (5) holds. It is not hard to conclude from (5) that there exists a unique isometry

$$\Theta^f : L^2(\widehat{G}, \mu_f) \longrightarrow L^2(X, m)$$

with $\Theta^f(\widehat{\sigma}) = N^f(\sigma)$ for any $\sigma \in C_c(G)$. Indeed, this can be seen as one version of the spectral theorem. If (X, G, m) is a TMDS, we can replace f by a canonical limiting object. More specifically, we can consider an approximate unit (φ_α) in $C_c(G)$ and define $f_\alpha \in L^2(X, m)$ by $f_\alpha(\mu) := \int \varphi_\alpha(-s) d\mu(s)$. Then,

$$N^{f_\alpha}(\varphi) \rightarrow N(\varphi) \text{ in } L^2(X, m).$$

Indeed, by construction and the defining properties of an approximate unit we have pointwise (in $\mu \in X$) convergence of $N^{f_\alpha}(\varphi)(\mu) = \varphi * (\varphi_\alpha * \mu)(0)$ to $\varphi * \mu(0) = N(\varphi)(\mu)$ as well as a uniform (in $\mu \in X$) bound $|N^{f_\alpha}(\varphi)(\mu)| \leq \|f\|_\infty |\mu|(K) \leq C < \infty$ due to the assumption on X . Thus, the maps N^{f_α} converge to the map N . Moreover, the spectral measures μ_{f_α} converge to $\widehat{\gamma}$ (see Corollary 1 in [3]). So, both N and $\widehat{\gamma}$ arise by a limiting procedure which involves an approximate unit.

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